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Hopf structure and Green ansatz of deformed parastatistics algebras

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Abstract

Deformed parabose and parafermi algebras are revised and endowed with Hopf structure in a natural way. The noncocommutative coproduct allows for construction of parastatistics Fock-like representations, built out of the simplest deformed Bose and Fermi representations. The construction gives rise to quadratic algebras of deformed anomalous commutation relations which define the generalized Green ansatz.

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1. Introduction

Wigner was the first to remark that the cannonical quantization was not the most general quantization scheme consistent with the Heisenberg equations of motions [1]. Parastatistics was introduced by Green [2] as a general quantization method of quantum field theory different from the cannonical Bose and Fermi quantization. This generalized statistics is based on two types of algebras with trilinear exchange relations, namely the parafermi and parabose algebras.

The representations of the parafermi and parabose algebras are labelled by a non-negative integer p—the order of parastatistics. The simplest non-trivial representations arise for $p = 1$ and coincide with the usual Bose (Fermi) Fock representations. The states in a Bose (Fermi) Fock space are totally symmetric (antisymmetric), i.e., they transform according to the one-dimensional representations of the *symmetric group*. Fock-like representations of parastatistics of order $p \geqslant 2$ correspond to higher dimensional representations of the symmetric group in the Hilbert space of multicomponent fields.

At the core of the interest in generalized statistics is (two-dimensional) statistical mechanics of phenomena such as the fractional Hall effect, high-T_c superconductivity. The experiments on the quantum Hall effect confirm the existence of fractionally charged excitations [3]. Models with fractional statistics and infinite statistics have been explored, termed as anyon statistics [4] and quon statistics [5].

The attempts to develop nonstandard quantum statistics evolved naturally to the study of deformed parastatistics algebras. The guiding principle in these developments is the isomorphism between the parabose algebra $\mathfrak{pB}(n)$, parafermi algebra $\mathfrak{pF}(n)$ (with n degrees of freedom) and the universal enveloping algebra of the orthosymplectic algebra $osp(1|2n)$, resp. orthogonal algebra $so(2n+1)$. The quantum counterparts $\mathfrak{pB}_q(n)$ and $\mathfrak{pF}_q(n)$ were defined to be isomorphic *as algebras* to the quantized universal enveloping algebras (QUEA) $U_q(osp(1|2n))$[6], resp. $U_q(so(2n+1))$ [7].

In the present work, we write a complete basis of defining relations of the algebras $\mathfrak{pB}_q(n)$ and $\mathfrak{pF}_q(n)$ (see theorem 1) extending what has been done in [6–8]. The novelty with respect to the known definition of deformed parastatistics is the system of homogeneous relations (9) and (10). They allow the isomorphism of the algebras to be continued as Hopf algebra morphism (see theorem 2) which endows the deformed parastatistics algebra at hand with natural Hopf structure. With the defined Hopf structure, the parastatistics algebras $\mathfrak{pB}_q(n)$ and $\mathfrak{pF}_q(n)$ become isomorphic *as Hopf algebras* to the QUEA $U_q(osp(1|2n))$ and $U_q(so(2n+1))$, respectively.

The Green ansatz is intimately related to the coproduct on the parastatistics algebras; it was realized that every parastatistics algebra representation of arbitrary order p arises through the iterated coproduct [9] (see also [10]). We make use of the noncocommutative coproduct on the Hopf parastatistics algebras $\mathfrak{pB}_q(n)$ and $\mathfrak{pF}_q(n)$ to construct a quadratic algebra which is a deformation of the Green ansatz for the classical algebras $\mathfrak{pB}(n)$ and $\mathfrak{pF}(n)$.

The paper is organized as follows. In section 2, we define the relations of the quantized parastatistics. Section 3 is devoted to the analysis of the Hopf algebra structure of the proposed quantized parastatistics algebras. In section 4, we show that the q-deformed bosonic (fermionic) oscillator algebra arises as the simplest non-trivial representation of the deformed parastatistics. Further, in section 5, the Green ansatz is generalized for the deformed parastatistics algebras $\mathfrak{pB}_q(n)$ and $\mathfrak{pF}_q(n)$.

Throughout the text by an associative algebra we mean an associative algebra with unit 1 over the complex numbers $\mathbb{C}$.

2. Deformed parastatistics algebras

We first recall the definitions of the parastatistics algebras introduced by Green [2] as a generalization of the Bose–Fermi alternative.

Definition 1. *The parafermi algebra $\mathfrak{pF}(n)$ (parabose algebra $\mathfrak{pB}(n)$) is an associative (super)algebra generated by the creation a_i^+ and annihilation a_i^- operators for $i=1,\ldots,n$, subject to the relations*

$$\begin{aligned} &[\![[\![a_i^+,a_j^-]\!],a_k^+]\!] = 2\delta_{jk}a_i^+, \qquad &&[\![[\![a_i^+,a_j^+]\!],a_k^+]\!] = 0,\\ &[\![[\![a_i^+,a_j^-]\!],a_k^-]\!] = -2\delta_{ik}a_j^-, \qquad &&[\![[\![a_i^-,a_j^-]\!],a_k^-]\!] = 0, \end{aligned} \tag{1}$$

where $[\![a,b]\!] = ab-(-1)^{\deg(a)\deg(b)}ba$ is the supercommutator and all the generators of $\mathfrak{pF}(n)$ ($\mathfrak{pB}(n)$) are taken to be even $\deg\left(a_j^\pm\right)=\bar{0}$ (*odd* $\deg(a_j^\pm)=\bar{1}$).[4]

The parafermi algebra is isomorphic to the universal enveloping algebra $U(so(2n+1))$ of the orthogonal algebra $so(2n+1), \mathfrak{pF}(n) \simeq U(so(2n+1))$ [11] while the parabose

[4] In this definition, only the linearly independent relations are written, other relations follow from the (super-)Jacobi identities.

algebra is isomorphic to the universal enveloping algebra $U(osp(1|2n))$ of the orthosymplectic superalgebra $osp(1|2n)$, $\mathfrak{pB}(n) \simeq U(osp(1|2n))$ [12].

The idea of quantization of the parastatistics algebras is to 'quantize' the classical isomorphisms, i.e., to deform the trilinear relations (1) in such a way that the arising deformed parafermi $\mathfrak{pF}_q(n)$ and parabose $\mathfrak{pB}_q(n)$ algebras are isomorphic to the *quantized universal enveloping algebra* (QUEA) of a Lie (super)algebra [13–16]

$$\mathfrak{pF}_q(n) \simeq U_q(so(2n+1)), \qquad \mathfrak{pB}_q \simeq U_q(osp(1|2n)). \tag{2}$$

The proofs of the algebra isomorphisms $\mathfrak{pB}_q \simeq U_q(osp(1|2n))$ [6] and $\mathfrak{pF}_q(n) \simeq U_q(so(2n+1))$ [7] have shown the equivalence of the paraoscillator definition of the $U_q(osp(1|2n))$ and $U_q(so(2n+1))$ with their definition in terms of Chevalley generators. In this way, a minimal set of relations (a counterpart of the Chevalley–Serre relations) is obtained providing an algebraic (but not linear) basis of the defining ideal of the QUEA at hand.

We are interested in a complete description of the defining ideal for the parastatistics algebras (i.e., the counterpart of the Cartan–Weyl definition of the QUEA). This is not only a question of pure academic interest, our motivation came from the study of the Hopf algebraic structure on the parastatistics algebras which to the best of our knowledge was studied only for some particular cases (see [17] for $\mathfrak{pB}_q(2)$). The complete basis of relations is generated from the known algebraic one and allows for endowing the $\mathfrak{pF}_q(n)$ and $\mathfrak{pB}_q(n)$ with a Hopf algebra structure. We now sketch the procedure of deriving the complete U_q-linear basis for the parastatistics algebras.

The Lie superalgebra $osp(1|2n)$, denoted as $B(0|n)$ in the Kac table [18], has the same Cartan matrix as the simple B_n algebra $so(2n+1)$. The Chevalley–Serre relations of QUEAs $U_q(so(2n+1))$ and $U_q(osp(1|2n))$ with generators $q^{\pm H_i} \equiv q^{\pm H_{\alpha_i}}$ and $E_{\pm i} \equiv E^{\pm\alpha_i}$, corresponding to the simple roots α_i, read

$$\begin{aligned}
&q^{H_i}q^{\pm H_j} = q^{\pm H_j}q^{H_i}, \qquad q^{H_i}E_{\pm j}q^{-H_i} = q^{\pm a_{ij}}E_{\pm j}, \qquad 1 \leqslant i, j \leqslant n,\\
&[2][E_i, E_{-j}] = \delta_{ij}[2H_i], \qquad [\![E_n, E_{-n}]\!] = [2H_n], \qquad 1 \leqslant i \leqslant n-1,\\
&[E_{\pm i}, E_{\pm j}] = 0, \qquad |i-j| \geqslant 2,\\
&[E_{\pm(i+1)}, [E_{\pm(i+1)}, E_{\pm i}]_q]_{q^{-1}} = 0, \qquad 1 \leqslant i \leqslant n-2,\\
&[E_{\pm i}, [E_{\pm i}, E_{\pm(i+1)}]_q]_{q^{-1}} = 0 = [[\![[E_{\pm(n-1)}, E_{\pm n}]_{q^{-1}}, E_{\pm n}]\!], E_{\pm n}]_q, \qquad 1 \leqslant i \leqslant n-1,
\end{aligned} \tag{3}$$

where $[x, y]_q = xy - qyx$ is the q-commutator, α_n is the only odd simple root of $osp(1|2n)$ and $a_{ij} = (\alpha_i, \alpha_j)$ is the symmetrized Cartan matrix (same for both cases) given by $a_{ij} = 2\delta_{ij} - \delta_{in} - \delta_{i+1j} - \delta_{ij+1}$. The quantum bracket is chosen to be $[x] := \frac{q^{\frac{x}{2}} - q^{-\frac{x}{2}}}{q^{\frac{1}{2}} - q^{-\frac{1}{2}}}$. The essential point in the proof of the isomorphism is the change of basis for the QUEA by choosing the orthogonal system of roots ε_i as an alternative to the simple roots. The ladder operators $E^{+\varepsilon_i}$ and $E^{-\varepsilon_i}$ related to the roots ε_i are the parastatistics creation and annihilation operators a_i^+ and a_j^- [6, 8] and the change $\varepsilon_i = \sum_{k=i}^n \alpha_k$ implies

$$\begin{aligned}
a_i^+ &= [E_i, [E_{i+1}, \ldots, [E_{n-1}, E_n]_{q^{-1}}, \ldots]_{q^{-1}}]_{q^{-1}},\\
a_i^- &= [[\ldots, [E_{-n}, E_{-n+1}]_q, \ldots, E_{-(i+1)}]_q, E_{-i}]_q
\end{aligned} \tag{4}$$

With the help of the inverse change $\alpha_i = \varepsilon_i - \varepsilon_{i+1}$, $i < n$, and $\alpha_n = \varepsilon_n$, the corresponding change of basis on the Cartan subalgebra reads $H_i = h_i - h_{i+1}$, $i < n$, and $H_n = h_n$. By construction, $q^{h_i}q^{h_j} = q^{h_j}q^{h_i}$. The inverse change of basis allows us to express the Chevalley ladder generators as

$$\begin{aligned}
&E_i = \frac{1}{[2]}q^{-h_{i+1}}[\![a_i^+, a_{i+1}^-]\!], \qquad E_{-i} = \frac{1}{[2]}[\![a_{i+1}^+, a_i^-]\!]q^{h_{i+1}}, \qquad i < n,\\
&E_n = a_n^+, \qquad E_{-n} = a_n^-.
\end{aligned} \tag{5}$$

The complete basis (over $\mathbb{C}(q)$) of relations defining the deformed parastatistics algebras, i.e., the analogue of (1) is given by the following:

Theorem 1. *The deformed parafermionic $\mathfrak{pF}_q(n)$ (parabosonic $\mathfrak{pB}_q(n)$) algebra is the associative (super)algebra generated by the creation and annihilation operators $a_i^\pm$ and Cartan generators $q^{\pm h_i}$ for $i = 1, \ldots, n$, subject to the relations*

$$q^{h_i} a_j^\pm q^{-h_i} = q^{\pm\delta_{ij}} a_j^\pm, \qquad [\![a_i^+, a_i^-]\!] = \frac{q^{h_i} - q^{-h_i}}{q^{\frac{1}{2}} - q^{-\frac{1}{2}}} = [2h_i], \tag{6}$$

$$[\![[\![a_i^+, a_j^-]\!], a_k^+]\!]_{q^{-\delta_{ik}\sigma_{j,k}}} = [2]\delta_{jk} a_i^+ q^{\sigma_{i,j} h_j} + (q - q^{-1})\theta_{i,j;k} a_i^+ [\![a_k^+, a_j^-]\!], \tag{7}$$

$$[\![[\![a_i^+, a_j^-]\!], a_k^-]\!]_{q^{-\delta_{jk}\sigma_{i,k}}} = -[2]\delta_{ik} a_j^- q^{-\sigma_{i,j} h_i} - (q - q^{-1})\theta_{j,i;k} [\![a_i^+, a_k^-]\!] a_j^-, \tag{8}$$

together with the analogues of the Serre relations

$$[\![[\![a_{i_1}^\pm, a_{i_3}^\pm]\!], a_{i_2}^\pm]\!]_{q^2} + q[\![[\![a_{i_1}^\pm, a_{i_2}^\pm]\!], a_{i_3}^\pm]\!] = 0, \qquad i_1 < i_2 \leqslant i_3, \tag{9}$$

$$[\![a_{i_2}^\pm, [\![a_{i_1}^\pm, a_{i_3}^\pm]\!]]\!]_{q^2} + q[\![a_{i_1}^\pm, [\![a_{i_2}^\pm, a_{i_3}^\pm]\!]]\!] = 0, \qquad i_1 \leqslant i_2 < i_3, \tag{10}$$

where all the generators $a_i^\pm$ are taken to be even, $\deg(a_i^\pm) = \bar{0}$ (odd, $\deg(a_i^\pm) = \bar{1}$) and the symbols $\theta_{i,j;k}, \sigma_{i,j}$ stay for $\theta_{i,j;k} = \frac{1}{2}\epsilon_{ij}\epsilon_{ijk}(\epsilon_{jk} - \epsilon_{ik})$, $\sigma_{i,j} = \epsilon_{ij} + \delta_{ij}$ or $\sigma_{i,j} = \epsilon_{ij} - \delta_{ij}$.[5]

The deformed parastatistics algebras admit an anti-involution $*$

$$(a_i^\pm)^* = a_i^\mp, \qquad (q^{\pm h_i})^* = q^{\mp h_i}, \qquad (q)^* = q^{-1} \tag{11}$$

induced by the anti-involution on the Chevalley basis $(E_{\pm i})^* = E_{\mp i}$, $H_i^* = H_i$.

To prove the theorem, we make use of the R-matrix FRT formalism for QUEA $U_q(g)$ of a simple (super-)Lie algebra g (see [15, 16]), introducing the L-functionals for $U_q(g)$ in the form of upper (lower)-triangular matrices $L^{(+)}$ $(L^{(-)})$

$$R^{(+)} L_1^{(\pm)} L_2^{(\pm)} = L_2^{(\pm)} L_1^{(\pm)} R^{(+)}, \qquad R^{(+)} L_1^{(+)} L_2^{(-)} = L_2^{(-)} L_1^{(+)} R^{(+)}, \tag{12}$$

where $L_1^{(\pm)} = L^{(\pm)} \otimes 1$, $L_2^{(\pm)} = 1 \otimes L^{(\pm)}$ and $R^{(+)} = \mathrm{PRP}$ is the corresponding R-matrix for $U_q(g)$.

The $(n+1) \times (n+1)$ minor $L_{ij}^{(+)}$, $1 \leqslant i, j \leqslant n+1$ of the $(2n+1) \times (2n+1)$ matrix $L^{(+)}$ for the QUEA $U_q(so(2n+1))$ and $U_q(osp(1|2n))$ is very simple when expressed in terms of the generators $a_i^\pm$

$$\left(L_{ij}^{(+)}\right)_{1\leqslant i,j\leqslant n+1} = \begin{pmatrix} q^{h_1} & \omega[\![a_1^+, a_2^-]\!] & \omega[\![a_1^+, a_3^-]\!] & \ldots & \omega[\![a_1^+, a_n^-]\!] & ca_1^+ \\ 0 & q^{h_2} & \omega[\![a_2^+, a_3^-]\!] & \ldots & \omega[\![a_2^+, a_n^-]\!] & ca_2^+ \\ 0 & 0 & q^{h_3} & \ldots & \omega[\![a_3^+, a_n^-]\!] & ca_3^+ \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \ldots & q^{h_n} & ca_n^+ \\ 0 & 0 & 0 & \ldots & 0 & 1 \end{pmatrix}, \tag{13}$$

where $\omega = q^{\frac{1}{2}} - q^{-\frac{1}{2}}$. The coefficient $c = q^{-\frac{1}{2}}(q - q^{-1})$. One has $\left(L_{ij}^{(+)}\right)^* = L_{ji}^{(-)}$.

[5] ϵ is the Levi-Civita symbol with $\epsilon_{ij} = 1$ for $i < j$. The tensor $\theta_{i,j;k} = -\theta_{j,i;k}$ is vanishing except for $i < k < j$ and $i > k > j$, when it takes values +1 and −1, respectively.

The relations (6)–(8) involving the entries of the minors of $L_{ij}^{(\pm)}$ (13) for $1 \leqslant i, j \leqslant n+1$ follow directly from the RLL relations (12) with the corresponding R-matrix upon restricting the indices from 1 to $n+1$. The restriction is possible due to the *ice condition* [19].

We label the lhs (up to scalars in $\mathbb{C}(q)$) of the homogeneous relations (9) and (10) by

$$\begin{aligned}
\Lambda_{i_2}^{i_1,i_3} &= [\![[\![a_{i_1}^+, a_{i_3}^+]\!], a_{i_2}^+]\!]_{q^2} + q[\![[\![a_{i_1}^+, a_{i_2}^+]\!], a_{i_3}^+]\!], \qquad \text{with} \quad i_1 < i_2 < i_3, \\
\Lambda_{i_2}^{i_1,i_2} &= [\![[\![a_{i_1}^+, a_{i_2}^+]\!], a_{i_2}^+]\!]_q, \qquad \text{with} \quad i_1 < i_2, \\
\tilde{\Lambda}_{i_3}^{i_1,i_2} &= [\![a_{i_2}^+, [\![a_{i_1}^+, a_{i_3}^+]\!]]\!]_{q^2} + q[\![a_{i_1}^+, [\![a_{i_2}^+, a_{i_3}^+]\!]]\!], \qquad \text{with} \quad i_1 < i_2 < i_3, \\
\tilde{\Lambda}_{i_3}^{i_2,i_2} &= [\![a_{i_2}^+, [\![a_{i_2}^+, a_{i_3}^+]\!]]\!]_q, \qquad \text{with} \quad i_2 < i_3.
\end{aligned} \tag{14}$$

The QUEA $U_q(gl_n)$ has a natural inclusion in $U_q(so(2n+1))$ and $U_q(osp(1|2n))$ being generated by the Chevalley generators $E_{\pm i}$, $1 \leqslant i \leqslant n-1$ and $q^{\pm h_i}$ (associated with the A_{n-1} subdiagram in the B_n Dynkin diagram). The inclusions $U_q(gl_n) \hookrightarrow \mathfrak{pF}_q(n)$ and $U_q(gl_n) \hookrightarrow \mathfrak{pB}_q(n)$ define an *adjoint* $U_q(gl_n)$ action on $\mathfrak{pF}_q(n)$ and $\mathfrak{pB}_q(n)$ (for $i \leqslant n-1$)

$$\mathrm{ad}_{E_i} a_j^+ = [E_i, a_j^+]_{q^{\delta_{ij}-\delta_{i+1j}}} = \delta_{i+1j} a_i^+, \qquad \mathrm{ad}_{E_{-i}} a_j^+ = [E_{-i}, a_j^+] q^{H_i} = \delta_{ij} a_{i+1}^+.$$

Let $\mathcal{L}$ denote the space of states Λ and $\tilde{\Lambda}$ where by states we mean the cubic polynomials determined from (14) up to multiplication with scalars $\mathbb{C}(q)$. The homogeneous relations (9) and (10) are $U_q(gl_n)$-covariant with respect to the adjoint action. More precisely, one has the following:

Lemma 1. *The space $\mathcal{L}$ is an irreducible finite-dimensional $U_q(gl_n)$ module with lowest weight. $\Lambda_n^{n-1,n}$ (for the proof see the appendix).*

The distinguished state $\Lambda_n^{n-1,n}$ expressed in terms of the Chevalley basis E_i is the last Serre relation in (3)

$$\Lambda_n^{n-1,n} = [[\![[E_{n-1}, E_n]_{q^{-1}}, E_n]\!], E_n]_q = 0$$

and thus $\Lambda_n^{n-1,n}$ has to be set to zero in $\mathfrak{pF}_q(n)$ and $\mathfrak{pB}_q(n)$. Hence, the whole representation $\mathcal{L}$ built through the $U_q(gl_n)$-adjoint action on the lowest weight $\Lambda_n^{n-1,n}$ is trivial which proves the homogeneous relations (9) and (10) for a_i^+. The ones for a_i^- follow by conjugation.

3. Hopf structure on parastatistics algebras

The QUE algebras $U_q(so(2n+1))$ and $U_q(osp(1|2n))$ (3) endowed with the Drinfeld–Jimbo coalgebraic structure [13, 14]

$$\begin{aligned}
\Delta H_i &= H_i \otimes 1 + 1 \otimes H_i, \qquad S(H_i) = -H_i, \qquad \epsilon(H_i) = 0, \\
\Delta E_i &= E_i \otimes 1 + q^{H_i} \otimes E_i, \qquad S(E_i) = -q^{-H_i} E_i, \qquad \epsilon(E_i) = 0, \\
\Delta E_{-i} &= E_{-i} \otimes q^{-H_i} + 1 \otimes E_{-i}, \qquad S(E_{-i}) = -E_{-i} q^{H_i}, \qquad \epsilon(E_{-i}) = 0
\end{aligned} \tag{15}$$

become Hopf algebra and Hopf superalgebra, respectively[6]. One has $S(x^*) = S(x)^*$.

The isomorphism (2) between the QUEA $U_q(so(2n+1))$ ($U_q(osp(1|2n))$) and the deformed parastatistics algebra induces a structure of a Hopf (super)algebra on $\mathfrak{pF}_q(n)$ ($\mathfrak{pB}_q(n)$). One can formulate the following:

Theorem 2. *The deformed parafermionic algebra $\mathfrak{pF}_q(n)$ (the deformed parabosonic algebra $\mathfrak{pB}_q(n)$) is a Hopf algebra (a Hopf superalgebra), when endowed with*

[6] For superalgebras, S is a graded antihomomorphism, $S(ab) = (-1)^{\deg(a)\deg(b)} S(b)S(a)$.

(i) a coproduct Δ defined on the generators by $\Delta q^{\pm h_i} = q^{\pm h_i} \otimes q^{\pm h_i}$,

$$\Delta a_i^+ = a_i^+ \otimes 1 + q^{h_i} \otimes a_i^+ + \omega \sum_{i<j\leqslant n} [\![a_i^+, a_j^-]\!] \otimes a_j^+, \tag{16}$$

$$\Delta a_i^- = a_i^- \otimes q^{-h_i} + 1 \otimes a_i^- - \omega \sum_{i<j\leqslant n} a_j^- \otimes [\![a_j^+, a_i^-]\!]; \tag{17}$$

(ii) a counit ϵ defined on the generators by $\epsilon(q^{\pm h_i}) = 1, \epsilon(a_i^\pm) = 0$;

(iii) an antipode S defined on the generators by $S(q^{\pm h_i}) = q^{\mp h_i}$,

$$S(a_i^+) = -q^{-h_i} a_i^+ - \sum_{s=1}^{n-i} (-\omega)^s \sum_{i<j_1<\cdots<j_s\leqslant n} W^{+i}{}_{j_1} W^{+j_1}{}_{j_2} \cdots W^{+j_{s-1}}{}_{j_s} q^{-h_{j_s}} a_{j_s}^+, \tag{18}$$

$$S(a_i^-) = \quad -a_i^- q^{h_i} - \sum_{s=1}^{n-i} (\omega)^s \sum_{n\geqslant j_s>\cdots>j_1>i} a_{j_s}^- q^{h_{j_s}} W^{-j_s}{}_{j_{s-1}} \cdots W^{-j_2}{}_{j_1} W^{-j_1}{}_{i}, \tag{19}$$

where $W^{+i}{}_j = q^{-h_i} [\![a_i^+, a_j^-]\!]$, $W^{-j}{}_i = [\![a_j^+, a_i^-]\!] q^{h_i}$ *and* $\omega = q^{\frac{1}{2}} - q^{-\frac{1}{2}}$.

Proof. The Hopf structure on the elements of $L^{(+)}$ and $L^{(-)}$ compatible with the Drinfeld structure (15) (defined on the Chevalley basis) is given by the coproduct $\Delta L^{\pm}$, the counit $\epsilon(L^{(\pm)})$ and the antipode $S(L^{(\pm)})$ [15]

$$\Delta L_{ik}^{(\pm)} = \sum_j L_{ij}^{(\pm)} \otimes L_{jk}^{(\pm)}, \qquad \epsilon(L_{ik}^{(\pm)}) = \delta_{ik}, \qquad \sum_j L_{ij}^{(\pm)} S(L_{jk}^{(\pm)}) = \delta_{ik}. \tag{20}$$

(i) For the diagonal elements $L_{ii}^{(+)} = q^{h_i}$, the coproduct formula in (20) yields

$$\Delta(L_{ii}^{(+)}) = \sum_{1\leqslant j\leqslant 2n+1} L_{ij}^{(+)} \otimes L_{ji}^{(+)} = L_{ii}^{(+)} \otimes L_{ii}^{(+)} = q^{\pm h_i} \otimes q^{\pm h_i}. \tag{21}$$

The coproduct of the elements $L_{i\,n+1}^{(+)}$ when $1 \leqslant i \leqslant n$ has the form

$$\Delta L_{i\,n+1}^{(+)} = \sum_{1\leqslant j\leqslant 2n+1} L_{ij}^{(+)} \otimes L_{j\,n+1}^{(+)} = L_{i\,n+1}^{(+)} \otimes 1 + \sum_{i\leqslant j\leqslant n} L_{ij}^{(+)} \otimes L_{j\,n+1}^{(+)}, \tag{22}$$

where we have used the triangularity of $L^{(+)}$ and $L_{n+1\,n+1}^{(+)} = 1$. Inserting into equation (22) the values $L_{i\,n+1}^{(+)} = ca_i^+$ (13) and abridging the constant c, we get

$$\Delta a_i^+ = a_i^+ \otimes 1 + \sum_{i\leqslant j\leqslant n} L_{ij}^{(+)} \otimes a_j^+, \tag{23}$$

which completes the proof of (16) in view of (13). Then, $\Delta a_i^- = (\Delta a_i^+)^*$.

(ii) It follows from the definition of the counit in (20).

(iii) For the diagonal elements the antipode formula in (20) implies $S(L_{ii}^{(+)}) = (L_{ii}^{(+)})^{-1}$, hence $S(q^{\pm h_i}) = q^{\mp h_i}$. For the nondiagonal elements due to the triangularity of $L^{(+)}$, the antipode formula (20) gives rise to the following system of equations:

$$\sum_{i\leqslant j\leqslant n+1} L_{ij}^{(+)} S(L_{j\,n+1}^{(+)}) = \delta_{i\,n+1} \quad \Longrightarrow \quad \sum_{i\leqslant j\leqslant n} L_{ij}^{(+)} S(L_{j\,n+1}^{(+)}) = -L_{i\,n+1}^{(+)}. \tag{24}$$

Here we have made use of $S\big(L^{(+)}_{n+1\,n+1}\big) = S(1) = 1$. In view of $S\big(L^{(+)}_{i\,n+1}\big) = cS\big(a_i^+\big)$, this is a linear triangular system for $S\big(a_i^+\big)$ which after normalization takes the form

$$S\big(a_i^+\big) + \omega \sum_{i<j\leqslant n} W_j^{i+} S\big(a_j^+\big) = -q^{-h_i} a_i^+, \qquad \text{where} \quad W_j^{i+} = q^{-h_i} [\![a_i^+, a_j^-]\!] \tag{25}$$

and the solution of this system yields equation (18). The antipodes $S\big(a_i^-\big)$ (19) are obtained through the conjugation, $S\big(a_i^-\big) = \big(S\big(a_i^+\big)\big)^*$. □

This theorem is interesting in its own right because it defines the Hopf structure on another basis of generators for QUEA of the algebra $so(2n+1)$ and the superalgebra $osp(1|2n)$.

4. The oscillator representations

The unitary representations π_p of the parastatistics algebras $\mathfrak{pB}(n)$ and $\mathfrak{pF}(n)$ (equation (1)) with unique vacuum state are indexed by a non-negative integer p [20] (see also [21] and references therein). The representation π_p is the lowest weight representation with a unique vacuum state $|0\rangle$ annihilated by all a_i^- and labelled by the *order of parastatistics* p

$$\pi_p\big(a_i^-\big)|0\rangle = 0, \qquad \pi_p\big(a_i^-\big)\pi_p\big(a_j^+\big)|0\rangle = p\delta_{ij}|0\rangle, \qquad \pi_0(x)|0\rangle = \epsilon(x)|0\rangle, \tag{26}$$

where the *vacuum representation*, i.e., the trivial one, corresponds to the counit ϵ of the Hopf parastatistics algebra. In the representation π_p (26) of the nondeformed parastatistics algebras (1), the Hamiltonian $\mathcal{H}_i = h_i = \frac{1}{2}\big[a_i^+, a_i^-\big]_\mp$ and the number operator $N_i = a_i^+ a_i^-$ associated with the ith paraoscillator are related by

$$\mathcal{H}_i = h_i = N_i \mp \frac{p}{2}, \tag{27}$$

where the upper (lower) sign is for parafermions (parabosons).

In the representation π_p of the deformed parastatistics algebras, the quantum analogue of the relation (27) holds

$$\big[a_i^+, a_i^-\big]_\mp = [2]\mathcal{H}_i, = [2h_i] = [2N_i \mp p],$$

which implies the deformed analogue of the π_p defining condition (26)

$$\pi_p\big(a_i^-\big)\pi_p\big(a_j^+\big)|0\rangle = [p]\delta_{ij}|0\rangle, \qquad \pi_0(x)|0\rangle = \epsilon(x)|0\rangle. \tag{28}$$

The constant $\mp[p]/[2]$ plays the role of energy of the vacuum as the constant $\mp p/2$ in (27) for the nondeformed algebras.

The algebra of the q-deformed fermionic (bosonic) oscillators $\mathfrak{F}_q(n)$ ($\mathfrak{B}_q(n)$) arises as a representation π of order $p = 1$ of the $\mathfrak{pF}_q(n)(\mathfrak{pB}_q(n))$

$$\left.\begin{array}{lll} \underline{a}_i^+\underline{a}_j^+ \pm q^{\mp\epsilon_{ij}}\underline{a}_j^+\underline{a}_i^+ = 0, & \underline{a}_i^-\underline{a}_j^- \pm q^{\mp\epsilon_{ij}}\underline{a}_j^-\underline{a}_i^- = 0 & \\ \underline{a}_i^-\underline{a}_i^+ \pm q\underline{a}_j^+\underline{a}_i^- = q^{\pm\underline{N}_i}, & \underline{a}_i^-\underline{a}_i^+ \pm q^{-1}\underline{a}_i^+\underline{a}_i^- = q^{\mp\underline{N}_i} & \\ \underline{a}_i^+\underline{a}_j^- \pm q^{\mp\epsilon_{ji}}\underline{a}_j^-\underline{a}_i^+ = 0, & \underline{a}_i^-\underline{a}_j^+ \pm q^{\mp\epsilon_{ji}}\underline{a}_j^+\underline{a}_i^- = 0, & i \neq j \end{array}\right\}. \tag{29}$$

We have adopted the notation $\pi(x) = \underline{x}$ and use $\underline{N}_i = \underline{h}_i \mp \frac{1}{2}$.

The analysis [22] of the positivity of the norm for the $\mathfrak{pB}_q(n)$ and $\mathfrak{pF}_q(n)$ representations in the simplest case $p = 1$ shows that such unitary representations (realized as finite-dimensional factor representations) exist only for q being a root of unity.

Remark. Unlike the case of $\mathfrak{pB}_q(n)$ and $\mathfrak{pF}_q(n)$, the deformed relations of bosonic and fermionic oscillator algebras (29) do not define Hopf ideals.

5. Green ansatz

The Green ansantz was introduced by Green in the same paper [2] in which he defined parastatistics. We briefly recall it and then bring it in a form convenient for deformation.

Let us consider a system with n degrees of freedom quantized in accordance with the parafermi or parabose statistics of order p, i.e., a system of n paraoscillators which is a particular representation π_p (of order p) of the parastatistics algebra with trilinear exchange relations (1).

The Green ansatz states that the parafermi (parabose) oscillators a_i^+ and a_i^- can be represented as sums of p Fermi (Bose) oscillators

$$\pi_p\left(a_i^{\pm}\right) = \sum_{r=1}^{p} a_i^{\pm(r)} \tag{30}$$

satisfying quadratic commutation relations of the same type (i.e., Fermi for parafermi and Bose for parabose) for equal indices (r)

$$\left[a_i^{-(r)}, a_k^{+(r)}\right]_{\pm} = \delta_{ik}, \qquad \left[a_i^{-(r)}, a_k^{-(r)}\right]_{\pm} = \left[a_i^{+(r)}, a_k^{+(r)}\right]_{\pm} = 0, \tag{31}$$

and of the opposite type for different indices

$$\left[a_i^{-(r)}, a_k^{-(s)}\right]_{\mp} = \left[a_i^{+(r)}, a_k^{+(s)}\right]_{\mp} = \left[a_i^{-(r)}, a_k^{+(s)}\right]_{\mp} = 0, \qquad r \neq s. \tag{32}$$

The upper (lower) signs stay for the parafermi (parabose) case.

The coproduct endows the tensor product of $\mathcal{A}$-modules of the Hopf algebra $\mathcal{A}$ with the structure of an $\mathcal{A}$-module. Thus, one can use the coproduct for constructing a representation out of simple ones. The simplest representations of the parastatistics algebras are the oscillator representations π (with $p = 1$). Higher representations π_p of parastatistics of order $p \geqslant 2$ arise through the iterated coproduct [9].

Let us denote the (p-fold) iteration of the coproduct by[7]

$$\Delta^{(0)} = \epsilon, \quad \Delta^{(1)} = \mathrm{id}, \quad \Delta^{(2)} = \Delta, \quad \ldots, \quad \Delta^{(p)} = (\underbrace{\Delta \otimes 1 \otimes \cdots \otimes 1}_{p-1}) \circ \Delta^{(p-1)} \tag{33}$$

and π denotes the projection from the (deformed) parafermi and parabose algebra onto the (deformed) fermionic $\mathfrak{F}$ ($\mathfrak{F}_q$) and bosonic $\mathfrak{B}$ ($\mathfrak{B}_q$) Fock representations, respectively.

Proposition 1. *The Green ansatz is equivalent to the commutativity of the following diagrams:*

$$\begin{array}{ccc} \mathfrak{pF}(n) & \xrightarrow{\Delta^{(p)}} & \mathfrak{pF}(n)^{\otimes p} \\ & \pi_p \searrow & \downarrow \pi^{\otimes p} \\ & & \mathfrak{F}(n)^{\otimes p} \end{array} \qquad \begin{array}{ccc} \mathfrak{pB}(n) & \xrightarrow{\Delta^{(p)}} & \mathfrak{pB}(n)^{\otimes p} \\ & \pi_p \searrow & \downarrow \pi^{\otimes p} \\ & & \mathfrak{B}(n)^{\otimes p} \end{array} \tag{34}$$

Proof. Using the coproduct of theorem 2 for $q = 1$ and projecting on the Fock representation we can choose the components of the Green ansatz to be the summands in the expressions

$$\pi^{\otimes p} \circ \Delta^{(p)}\left(a_i^{\pm}\right) = \sum_{r=1}^{p} \underbrace{1 \otimes \cdots \otimes 1}_{r-1} \otimes \pi\left(a_i^{\pm}\right) \otimes \underbrace{1 \otimes \cdots \otimes 1}_{p-r} := \sum_{r=1}^{p} a_i^{\pm(r)}. \tag{35}$$

The check that the Green components $a_i^{\pm(r)}$ satisfy the bilinear commutation relations (31) and (32) is direct, however, one has to keep in mind that the tensor product is $\mathbb{Z}_2$-graded in the parabose case and non-graded in the parafermi case, which explains why the anomalous commutation relations (32) appear. We emphasize that the grading of the tensor product turns

[7] The definition of $\Delta^{(p)}$ extended with the counit ϵ is consistent with $\pi_0 = \epsilon$.

out to be the opposite to the (independent) grading of the Bose or Fermi algebra which appears on each site (r).

The diagrams (34) are commutative if and only if

$$\pi_p\left(a_i^{\pm}\right) = \pi^{\otimes p} \circ \Delta^{(p)}\left(a_i^{\pm}\right), \tag{36}$$

which is exactly the statement of the Green ansatz (30). □

We are now in a position to extend the Green ansatz to the deformed parafermi $\mathfrak{pF}_q(n)$ and parabose $\mathfrak{pB}_q(n)$ algebras. The simplest representations of $\mathfrak{pF}_q(n)$ and $\mathfrak{pB}_q(n)$ of parastatistics order $p = 1$ are the deformed fermionic $\mathfrak{F}_q$ and bosonic $\mathfrak{B}_q$ Fock representations, respectively, and let π be the projection on these Fock spaces.

Definition 2. *The system of quadratic exchange relations stemming from the commutativity of the diagrams*

$$\begin{array}{ccc} \mathfrak{pF}_q(n) & \xrightarrow{\Delta^{(p)}} & \mathfrak{pF}_q(n)^{\otimes p} \\ & \pi_p \searrow & \downarrow \pi^{\otimes p} \\ & & \mathfrak{F}_q(n)^{\otimes p} \end{array}, \qquad \begin{array}{ccc} \mathfrak{pB}_q(n) & \xrightarrow{\Delta^{(p)}} & \mathfrak{pB}_q(n)^{\otimes p} \\ & \pi_p \searrow & \downarrow \pi^{\otimes p} \\ & & \mathfrak{B}_q(n)^{\otimes p} \end{array} \tag{37}$$

is the deformed Green ansatz of parastatistics of order p. Here $\Delta^{(p)}$ stays for the p-fold noncocommutative coproduct (33) on the Hopf algebras $\mathfrak{pF}_q(n)$ and $\mathfrak{pB}_q(n)$ (see theorem 2).

Let us show the consistency of condition (28) with the deformed Green ansatz. The vacuum state $|0\rangle^{(p)}$ of the representation π_p is to be identified with the tensor power of the oscillator ($p = 1$) vacuum, $|0\rangle^{(p)} = |0\rangle^{\otimes p}$. Evaluating the iterated graded commutator (6)

$$\Delta^{(p)}[\![a_i^+, a_i^-]\!] = [\![\Delta^{(p)}a_i^+, \Delta^{(p)}a_i^-]\!] = \frac{(q^{h_i})^{\otimes p} - (q^{-h_i})^{\otimes p}}{q^{\frac{1}{2}} - q^{-\frac{1}{2}}} \tag{38}$$

on the vacuum state $|0\rangle^{\otimes p}$ in the oscillator representations $\pi^{\otimes p}$, we get the defining condition (28) of the deformed π_p

$$\begin{aligned} \mp\pi^{\otimes p} \circ \Delta^{(p)}[\![a_i^+, a_i^-]\!]\,|0\rangle^{(p)} &= \pi_p(a_i^-)\pi_p\left(a_i^+\right)|0\rangle^{(p)} \\ &= [p]\,|0\rangle^{(p)}, \qquad \left(= \frac{q^{\frac{p}{2}} - q^{-\frac{p}{2}}}{q^{\frac{1}{2}} - q^{-\frac{1}{2}}}\,|0\rangle^{(p)}\right), \end{aligned}$$

since $\pi(q^{h_i}) = q^{N_i \mp \frac{1}{2}}$, which proves the consistency.

The Green components $a_i^{\pm(r)}$ in a $\mathfrak{pF}_q(n)$ or $\mathfrak{pB}_q(n)$ representation π_p of parastatistics of order p will be chosen to be

$$\begin{aligned} a_i^{+(r)} &= \pi^{\otimes p} \circ \Delta^{(r-1)} \otimes 1 \otimes \Delta^{(p-r)}\left(\sum_{k=1}^{n} L_{ik}^{(+)} \otimes a_k^+ \otimes 1\right), \\ a_i^{-(r)} &= \pi^{\otimes p} \circ \Delta^{(r-1)} \otimes 1 \otimes \Delta^{(p-r)}\left(\sum_{k=1}^{n} 1 \otimes a_k^- \otimes L_{ki}^{(-)}\right). \end{aligned} \tag{39}$$

Note that the conjugation $*$ acts as a reflection on the Green indices (r)

$$\left(a_i^{\pm(r)}\right)^* = a_i^{\mp(r^*)}, \qquad r^* = p - r + 1.$$

More explicitly, the Green components look like

$$\begin{aligned} a_i^{+(r)} &= \sum_{k_1,\ldots,k_r} \underline{L}_{ik_1}^{(+)} \otimes \underline{L}_{k_1k_2}^{(+)} \otimes \cdots \otimes \underline{L}_{k_{r-1}k_r}^{(+)} \otimes \underline{a}_{k_r}^+ \otimes 1 \cdots \otimes 1, \\ a_j^{-(r)} &= \sum_{k_1,\ldots,k_{p-r}} 1 \otimes \cdots \otimes 1 \otimes \underline{a}_{k_1}^- \otimes \underline{L}_{k_1k_2}^{(-)} \otimes \underline{L}_{k_2k_3}^{(-)} \otimes \cdots \otimes \underline{L}_{k_{p-r}j}^{(-)}, \end{aligned} \tag{40}$$

where the upper (lower) triangularity of the matrices $L^{(+)}(L^{(-)})$ infers that only the terms subject to the inequalities $i \leqslant k_1 \leqslant \cdots \leqslant k_r \leqslant n$ are non-zero (respectively, $n \geqslant k_1 \geqslant \cdots \geqslant k_{p-r} \geqslant j$). Unlike the non-deformed case, each Green component $a_i^{\pm(r)}$ in the deformed Green ansatz is a sum of many terms resulting from the mapping $\pi^{\otimes p} \circ \Delta^{(p)}$.

To present the results in a more concise form, we introduce the operators

$$Q_{ji}^{+(r)} = \pi^{\otimes p} \circ \Delta^{(r)} \otimes \Delta^{(p-r)} \left(\sum_{k=1}^{n} L_{jk}^{(+)} \otimes L_{ki}^{(-)} \right), \tag{41}$$

$$Q_{ji}^{-(r)} = \pi^{\otimes p} \circ \Delta^{(r-1)} \otimes \Delta^{(p-r+1)} \left(\sum_{k=1}^{n} L_{jk}^{(+)} \otimes L_{ki}^{(-)} \right). \tag{42}$$

One readily sees that $\left(Q_{ji}^{+(r)}\right)^* = Q_{ij}^{-(r^*)}$ and $\left(Q_{ji}^{-(r)}\right)^* = Q_{ij}^{+(r^*)}$.

We now summarize the deformed quadratic algebra anomalous commutation rules. For different Green indices, the Green components (40) quommute ($[x, y]_{\pm q} = xy \pm qyx$) as follows (we suppose $r > s$):

$$\begin{aligned} &\left[a_i^{+(r)}, a_j^{+(s)}\right]_{\mp} = \mp(q - q^{-1}) a_j^{+(r)} a_i^{+(s)}, \qquad && \left[a_i^{-(r)}, a_j^{-(s)}\right]_{\mp} = 0, \qquad && i < j, \\ &\left[a_i^{-(r)}, a_j^{-(s)}\right]_{\mp} = \pm(q - q^{-1}) a_j^{-(r)} a_i^{-(s)}, && \left[a_i^{+(r)}, a_j^{+(s)}\right]_{\mp} = 0, && i > j, \end{aligned} \tag{43}$$

$$\left[a_i^{+(r)}, a_i^{+(s)}\right]_{\mp q} = 0, \qquad \left[a_i^{-(r)}, a_i^{-(s)}\right]_{\mp q^{-1}} = 0, \qquad \left[a_i^{-(r)}, a_j^{+(s)}\right]_{\mp} = 0. \tag{44}$$

When the Green indices coincide, one gets

$$\begin{aligned} &\left[a_i^{+(r)}, a_j^{+(r)}\right]_{\pm q^{\mp\epsilon_{ij}}} = 0, \qquad && \left[a_i^{-(r)}, a_j^{-(r)}\right]_{\pm q^{\mp\epsilon_{ij}}} = 0, \\ &\left[a_i^{-(r)}, a_j^{+(r)}\right]_{\pm q^{\mp 1}} = q^{\mp\frac{1}{2}} Q_{ji}^{-(r)}, && \left[a_i^{-(r)}, a_j^{+(r)}\right]_{\pm q^{\pm 1}} = q^{\pm\frac{1}{2}} Q_{ji}^{+(r)}, \end{aligned} \tag{45}$$

where the operators $Q_{ji}^{+(r)}$ and $Q_{ji}^{-(r)}$ (42) are quadratic in the Green components

$$\begin{aligned} q^{\mp\frac{1}{2}} Q_{ji}^{-(r)} &= (q - q^{-1}) \sum_{s=1}^{r-1} q^{\mp(r-s)} a_j^{+(s)} a_i^{-(s)} = \left(q^{\pm\frac{1}{2}} Q_{ij}^{+(r^*)}\right)^*, \qquad i > j, \\ q^{\mp\frac{1}{2}} Q_{ji}^{-(r)} &= -(q - q^{-1}) \sum_{s=r}^{p} q^{\mp(r-s)} a_j^{+(s)} a_i^{-(s)} = \left(q^{\pm\frac{1}{2}} Q_{ij}^{+(r^*)}\right)^*, \qquad i < j, \end{aligned} \tag{46}$$

$$\begin{aligned} q^{\pm\frac{1}{2}} Q_{ii}^{+(r)} &= q^{\mp(r-\frac{p}{2}-\frac{1}{2})} (q^{N_i})^{\otimes p} - (q - q^{-1}) \sum_{s=r+1}^{p} q^{\mp(r-s)} a_i^{+(s)} a_i^{-(s)}, \\ q^{\mp\frac{1}{2}} Q_{ii}^{-(r)} &= q^{\mp(r-\frac{p}{2}-\frac{1}{2})} (q^{-N_i})^{\otimes p} + (q - q^{-1}) \sum_{s=1}^{r-1} q^{\mp(r-s)} a_i^{+(s)} a_i^{-(s)}. \end{aligned} \tag{47}$$

The system of relations (43)–(47) with the upper (lower) signs defines the generalization of the Green ansatz for the deformed parafermi $\mathfrak{pF}_q(n)$ (parabose $\mathfrak{pB}_q(n)$) algebras.

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Appendix

Proof of lemma 1. All lowering $U_q(gl_n)$ Chevalley generators E_{-i} kill the state $\Lambda_n^{n-1,n}$

$$\mathrm{ad}_{E_{-i}}\Lambda_n^{n-1,n} = 0, \qquad i = 1, \ldots, n-1.$$

The states $\Lambda_j^{i,n}$ and $\tilde{\Lambda}_n^{i,j}$ for all admissible i, j (14) arise through the adjoint action of the raising $U_q(gl_n)$ generators as seen from the diagram in which the decorated arrows denote the adjoint actions ad_{E_i}

$$\begin{tikzcd}
\Lambda_n^{n-1,n} \arrow[r, "E_{n-1}"] \arrow[d, "E_{n-2}"] & \tilde{\Lambda}_n^{n-1,n-1} \arrow[r, "E_{n-2}"] & \tilde{\Lambda}_n^{n-2,n-1} \arrow[r, "E_{n-3}"] \arrow[d, "E_{n-2}"] & \cdots \arrow[r, "E_2"] & \tilde{\Lambda}_n^{2,n-1} \arrow[r, "E_1"] \arrow[d, "E_{n-2}"] & \tilde{\Lambda}_n^{1,n-1} \arrow[d, "E_{n-2}"] \\
\Lambda_n^{n-2,n} \arrow[r, "E_{n-1}"] \arrow[d, "E_{n-3}"] & \Lambda_{n-1}^{n-2,n} \arrow[r, "E_{n-2}"] \arrow[d, "E_{n-3}"] & \tilde{\Lambda}_n^{n-2,n-2} \arrow[r, "E_{n-3}"] & \cdots \arrow[r, "E_2"] & \tilde{\Lambda}_n^{2,n-2} \arrow[r, "E_1"] \arrow[d, "E_{n-3}"] & \tilde{\Lambda}_n^{1,n-2} \arrow[d, "E_{n-3}"] \\
\Lambda_n^{n-3,n} \arrow[r, "E_{n-1}"] \arrow[d, "E_{n-4}"] & \Lambda_{n-1}^{n-3,n} \arrow[r, "E_{n-2}"] \arrow[d, "E_{n-4}"] & \Lambda_{n-2}^{n-3,n} \arrow[r, "E_{n-3}"] \arrow[d, "E_{n-4}"] & \cdots \arrow[r, "E_2"] & \tilde{\Lambda}_n^{2,n-3} \arrow[r, "E_1"] \arrow[d, "E_{n-4}"] & \tilde{\Lambda}_n^{1,n-3} \arrow[d, "E_{n-4}"] \\
\vdots \arrow[d, "E_1"] & \vdots \arrow[d, "E_1"] & \vdots \arrow[d, "E_1"] & & \vdots & \vdots \arrow[d, "E_1"] \\
\Lambda_n^{1,n} \arrow[r, "E_{n-1}"] & \Lambda_{n-1}^{1,n} \arrow[r, "E_{n-2}"] & \Lambda_{n-2}^{1,n} \arrow[r, "E_{n-3}"] & \cdots \arrow[r, "E_2"] & \Lambda_2^{1,n} \arrow[r, "E_1"] & \tilde{\Lambda}_n^{1,1}
\end{tikzcd}.$$

Next, the new state $\Lambda_{n-1}^{n-2,n-1} = \mathrm{ad}_{E_{n-1}}\Lambda_{n-1}^{n-2,n}$ stays at the top of a new diagram of the same type with $n' = n-1$. By induction, we obtain all the states in $\mathcal{L}$ (14). The state $\tilde{\Lambda}_2^{1,1}$ is the highest weight of $\mathcal{L}$. One can check that the adjoint $U_q(gl_n)$ action does not bring out of $\mathcal{L}$ which completes the proof. $\square$

The $U_q(gl_n)$-module $\mathcal{L}$ is a smooth deformation of a Schur module associated with the Young diagram $\lambda = (2, 1)$ [23]. The states $\Lambda_j^{i,k}$ and $\tilde{\Lambda}_k^{i,j}$ in $\mathcal{L}$ are labelled with semistandard Young tableaux. Hence, the dimension is $\dim\mathcal{L} = \frac{(n+1)n(n-1)}{3}$.

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